International Journal of Thermal Sciences 49 (2010) 67–75

Contents lists available at ScienceDirect

International Journal of Thermal Sciences

journal homepage: www.elsevier.com/locate/ijts

Elastic effects on Rayleigh-Bénard convection in liquids with temperature-dependent viscosity

G.N. Sekhar [a], G. Jayalatha [b,*]

[a] Department of Mathematics, BMS College of Engineering, Bangalore 560019, India
[b] Department of Mathematics, RV College of Engineering, Bangalore 560059, India

ARTICLE INFO

Article history:
Received 17 March 2009
Received in revised form
5 June 2009
Accepted 11 June 2009
Available online 17 July 2009

Keywords:
Convection
Rayleigh-Bénard convection
Viscoelastic liquids
Variable viscosity
Thermorheology

ABSTRACT

A linear stability analysis of convection in viscoelastic liquids with temperature-dependent viscosity is studied using normal modes and Galerkin method. Stationary convection is shown to be the preferred mode of instability when the ratio of strain retardation parameter to stress relaxation parameter is greater than unity. When the ratio is less than unity then the possibility of oscillatory convection is shown to arise. Oscillatory convection is studied numerically for Rivlin-Ericksen, Maxwell and Jeffreys liquids by considering free-free, rigid-rigid and rigid-free isothermal/adiabatic boundaries. The effect of variable viscosity parameter is shown to destabilize the system. The problem reveals the stabilizing nature of strain retardation parameter and destabilizing nature of stress relaxation parameter, on the onset of convection. The Maxwell liquids are found to be more unstable than the one subscribing to Jeffreys description whereas the Rivlin-Ericksen liquid is comparatively more stable. Free-free adiabatic boundary combination is found to give rise to a most unstable system, whereas the rigid isothermal rigid adiabatic combination gives rise to a most stable system. The problem has applications in non-isothermal systems having viscoelastic liquids as working media.

1. Introduction

Rayleigh-Bénard convection problems in constant viscosity Newtonian liquids have received widespread attention due to their implications in heat transfer and in other engineering applications (see Chandrasekhar [1]; Platten and Legros [2]). In practical problems, many a time, non-Newtonian liquids are used as working media, especially the viscoelastic ones. In view of this several works have appeared on Rayleigh-Bénard convection in these liquids.

Herbert [3] examined the stability of plane Couette flow heated from below. He showed that finite elastic stress in the undisturbed state is necessary for the oscillatory motion. He also showed that presence of elasticity has a destabilizing effect on the flow. Green [4] studied oscillatory convection in an elasticoviscous liquid. He found that a large restoring force sets up an oscillating convective motion in a thin rectangular layer of the fluid heated from below.

Vest and Arpaci [5] have studied the conditions under which thermally induced overstability occurs in a viscoelastic liquid. It is found that overstability would occur at the lowest possible adverse temperature gradient at which the rate of change of kinetic energy can balance in a synchronous manner, the periodically varying rates of energy dissipation by the shear stresses and the energy release by the buoyancy force, assuming that stationary convection has not been initialized. It is also found that oscillatory convection occurs at a lower critical Rayleigh number than stationary convection.

Sokolov and Tanner [6] have studied Rayleigh-Bénard convection in a general viscoelastic fluid using integral form of constitutive equations. It is shown that under certain conditions the fluid system is overstable. The theoretical results have been applied to a Maxwell fluid and to some real viscoelastic solutions. It is found that very high temperature gradients or high gravitational fields would be required before the oscillating cells could be observed in common polymer solutions of moderate viscosity.

Riahi [7] has studied nonlinear convection in viscoelastic fluids using boundary layer method, assuming large Rayleigh number, Prandtl number and a small value of elasticity parameter. It is shown that elasticity effects do not affect the horizontal wave number significantly and also that the heat flux depends strongly on elasticity parameter and decreases with increasing elasticity parameter. Eltayeb [8] has studied linear and nonlinear Rayleigh-Bénard convection in a viscoelastic fluid using Oldroyd model. It is

* Corresponding author. Present address: Department of Mathematics, BMS College of Engineering, Bangalore 560019, India.
E-mail addresses: drgns.maths@bmsce.ac.in (G.N. Sekhar), jayalatha.maths@bmsce.ac.in (G. Jayalatha).

1290-0729/$ – see front matter © 2009 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.ijthermalsci.2009.06.003

Nomenclature

a	dimensionless wave number
d	thickness of the liquid
g_i	components of gravitational acceleration (0, 0, $-g$)
l, m	wave numbers
p	effective pressure
Pr	Prandtl number
q_i	velocity components (u, v, w)
Q	elastic ratio (λ_2/λ_1) or scaled strain retardation parameter
R	Rayleigh number
t	time
T	temperature
T_0	constant temperature of the upper boundary
T_a	average of boundary temperatures $(T_0+T_0+\Delta T)/2$
V	variable viscosity parameter

Greek symbols

α	thermal expansion coefficient
δ_i	small positive constants
κ	thermal diffusivity
λ_1	stress relaxation coefficient
λ_2	strain retardation coefficient
Λ_1	Deborah number or scaled stress relaxation parameter
μ	temperature-dependent viscosity
ω	frequency
ρ	density
ρ_0	reference density

Subscripts

a	average quantity
b	basic state
c	critical quantity
oc	oscillatory critical quantity
s	stationary quantity
sc	stationary critical quantity

Superscripts

$'$	dimensional quantity
*	dimensionless quantity

found that, in the study of nonlinear effects for slightly supercritical Rayleigh number, the plane disturbances for the case where the exchange of stabilities is valid and plane disturbances for the case of overstability are governed by equations similar to that for the plane Poiseulle flow. He also showed that elasticity effect is to stabilize the layer in the linear theory and to destabilize it in the nonlinear theory provided the ratio of the mean temperature gradient of the layer to the actual temperature difference across the layer is large enough.

Shenoy and Mashelkar [9] have considered thermal convection in non-Newtonian fluids. They studied overstability that emerges in practical situations such as in geophysical applications. It is found that the onset of thermal convection, during the flow of polycrystalline rock in the earth's mantle, occurs in the presence of high strain rates, which in turn make the viscosity strain rate dependent. Rosenblat [10] has studied thermal convection in a viscoelastic liquid, employing a general constitutive relation. He analyzed the Rayleigh-Bénard convection of viscoelastic liquids for free boundary conditions whose eigen functions are easily obtained analytically. Employing the power series method for a nonlinear stability analysis he also showed that Hopf bifurcations and exchange of stabilities are possible and revealed that subcritical bifurcation is permissible in addition to the usual supercritical bifurcation, depending on the values of the viscoelastic parameters. Kolkka and Ierley [11] have considered the convected linear stability of a viscoelastic Oldroyd-B fluid heated from below. They extended the linear analysis to the case of a rigid boundary by using a numerical method and one term Galerkin approximation for a constitutive relation of Oldroyd type.

Khayat [12] has examined the onset of aperiodic or chaotic behavior in viscoelastic fluids, namely Oldroyd-B fluid in the context of the Rayleigh-Bénard convection problem. It is found that fluid elasticity tends to destabilize the convective cell structure, precipitating the onset of chaotic motion, at a Rayleigh number that may be well below that corresponding to Newtonian fluids. Khayat [13] has examined the existence and stability of elastic overstability in the presence of non-negligible inertia for an Oldroyd-B fluid for the Rayleigh-Bénard convection problem. By applying center manifold theory, it is shown that the Hopf bifurcation corresponding to the onset of the periodic orbit exists and is asymptotically stable to small perturbations of the conductive state for highly elastic flows.

Park and Lee [14] have analyzed Hopf bifurcation of viscoelastic fluids heated from below for realistic boundaries such as rigid-rigid and rigid-free cases for the range of viscoelastic parameters where Hopf bifurcation occurs. The nonlinear stability analysis using the power series method reveal that values of viscoelastic parameters have significant effect on hydrodynamic stability and suggest that the system of Rayleigh-Bénard convection may be used at least in part as a useful rheometric tool to assess the suitability of the constitutive equations.

Mardones *et al.* [15] have studied thermal convection thresholds in viscoelastic solutions. The threshold for oscillatory motions in Rayleigh-Bénard experiments with viscoelastic binary fluids is explicitly determined as a function of separation ratio and rheological parameters. It is shown that the critical oscillation frequency may differ by several orders of magnitude on varying separation ratio and Deborah number. The results suggest that binary fluid aspect ratio may not be discarded when studying thermal convection in polymeric solutions. It is concluded that oscillatory convection in viscoelastic solutions is very sensitive to actual values of separation ratio.

Siddheshwar and Srikrishna [16] have studied effect of nonuniform temperature gradient on the linear stability analysis of

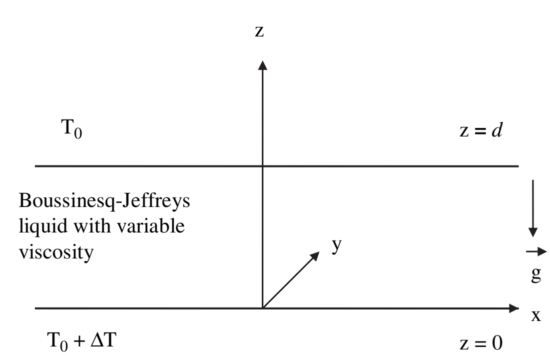

Fig. 1. Schematic of the problem.

Table 1
Different boundary combinations and corresponding trial functions.

Case	Boundary	Boundary condition (BC)	Acronym for boundary condition	Trial functions
1	$z=0$	$w=D^2$, $w=0$, free $T=0$, isothermal	FIFI	$w_1=z^4-2z^3+z$, $T_1=z^2-z$. (Symmetric BC)
	$z=1$	$w=D^2$, $w=0$, free $T=0$, isothermal		
2	$z=0$	$w=D$, $w=0$, rigid $T=0$, isothermal	RIFI	$w_1=2z^4-5z^3+3z^2$, $T_1=z^2-z$.
	$z=1$	$w=D^2$, $w=0$, free $T=0$, isothermal		
3	$z=0$	$w=D$, $w=0$, rigid $T=0$, isothermal	RIRI	$w_1=z^4-2z^3+z^2$, $T_1=z^2-z$. (Symmetric BC)
	$z=1$	$w=D$, $w=0$, rigid $T=0$, isothermal		
4	$z=0$	$w=D$, $w=0$, rigid $DT=0$, adiabatic	RAFI	$w_1=2z^4-5z^3+3z^2$, $T_1=z^2-1$.
	$z=1$	$w=D^2$, $w=0$, free $T=0$, isothermal		
5	$z=0$	$w=D$, $w=0$, rigid $T=0$, isothermal	RIRA	$w_1=z^4-2z^3+z^2$, $T_1=z^2-2z$.
	$z=1$	$w=D$, $w=0$, rigid $DT=0$, adiabatic		
6	$z=0$	$w=D$, $w=0$, free $DT=0$, adiabatic	FAFI	$w_1=z^4-2z^3+z$, $T_1=z^2-1$.
	$z=1$	$w=D$, $w=0$, free $T=0$, isothermal		
7	$z=0$	$w=D$, $w=0$, rigid $T=0$, isothermal	RIFA	$w_1=2z^4-5z^3+3z^2$, $T_1=z^2-2z$.
	$z=1$	$w=D^2$, $w=0$, free $DT=0$, adiabatic		
8	$z=0$	$w=D^2$, $w=0$, free $DT=0$, adiabatic	FAFA	$w_1=z^4-2z^3+z$, $T_1=\cos \pi z$. (Symmetric BC)
	$z=1$	$w=D^2$, $w=0$, free $DT=0$, adiabatic		
9	$z=0$	$w=D$, $w=0$, rigid $DT=0$, adiabatic	RAFA	$w_1=2z^4-5z^3+3z^2$, $T_1=\cos \pi z$.
	$z=1$	$w=D^2$, $w=0$, free $DT=0$, adiabatic		
10	$z=0$	$w=D$, $w=0$, rigid $DT=0$, adiabatic	RARA	$w_1=z^4-2z^3+z^2$, $T_1=\cos \pi z$. (Symmetric BC)
	$z=1$	$w=D$, $w=0$, rigid $DT=0$, adiabatic		

the Rayleigh-Bénard convection problem in a viscoelastic fluid-filled high-porosity medium using single-term Galerkin technique. It is found that the strain retardation time should be less than the stress relaxation time for oscillatory convection to set in, in a high-porosity medium. The analysis also predicts the critical eigen value for the viscoelastic problem to be less than that of the corresponding Newtonian fluid problem. Park and Ryu [17] have made a linear stability analysis of the Rayleigh-Bénard convection problem in viscoelastic fluids, in finite domains. A Chebyshev pseudospectral method is generalized to solve the hydrodynamic stability problem. A very general constitutive equation that encompasses the Maxwell model, the Oldroyd model and the Phan-Thien-Tanner model is adopted. It is found that the results may be used to investigate the appropriateness of a constitutive equation and its parameter values adopted for a given viscoelastic fluid.

Siddheshwar [18] has studied oscillatory convection in viscoelastic ferromagnetic and dielectric liquids of the Rivlin-Ericksen, Maxwell and Oldroyd types. It is found that the Maxwell liquids are more unstable than the one subscribing to the Oldroyd description whereas the Rivlin-Ericksen liquid is comparatively more stable. Siddheshwar and Srikrishna [19] have made linear and nonlinear stability analyses of convection in a second order fluid, described by the Rivlin-Ericksen constitutive equation. The linear theory based on normal mode technique reveals that critical eigen value is independent of viscoelastic effects. The nonlinear analysis based on truncated representation of Fourier series reveals that finite amplitudes have random behavior. The onset of chaotic motion is also discussed.

Abu-Ramadan *et al.* [20] have studied chaotic thermal convection of viscoelastic fluids. The viscoelastic flow in the context of the Rayleigh-Bénard thermal convection set-up is examined using a four-dimensional nonlinear dynamical system resulting from a truncated Fourier representation of the conservation and constitutive equations, for an Oldroyd-B fluid. The routes to chaos are identified and the dynamical response of the flow with change of control parameters is illustrated. Fluid elasticity and fluid retardation were found to alter the flow behavior and elasticity was found to precipitate the onset of chaos.

Kim *et al.* [21] have studied thermal instability of viscoelastic fluids in porous media. A theoretical analysis of thermal instability driven by buoyancy forces in an initially quiescent, horizontal porous layer saturated by viscoelastic fluids, neglecting the difference between the heat capacity of the fluid and that of the porous matrix is conducted.

Park and Park [22] have considered Rayleigh-Bénard convection of viscoelastic fluids in arbitrary finite domains. It is shown that the domain shape can change the viscoelastic parameter values where Hopf bifurcation occurs in the Rayleigh-Bénard convection. The effects of the shape of the domain and rheological parameter values on the critical Rayleigh number and convection pattern are examined. Yoon *et al.* [23] analytically analyzed the onset of thermal convection in an isothermally heated, horizontal porous layer saturated with viscoelastic liquid, using linear theory. It is shown that oscillatory instabilities set in before stationary modes are exhibited. Bertola and Cafaro [24] have examined thermal instability of viscoelastic fluids in horizontal porous layers. The viscoelastic character of the flow is taken into account by a modified Darcy's law. Qualitative expressions of the Nusselt number when the system is out of equilibrium are derived.

Siddheshwar and Srikrishna [25] have considered thermal instability in a layer of dilute polymeric liquids when boundaries are subjected to imposed time periodic boundary temperatures. Periodicity and amplitude criterion for thermal instability is determined. The qualitative effects of various governing parameters on the convective system are discussed.

The above works address the problem of Rayleigh-Bénard convection in viscoelastic liquids with constant viscosity. It is well known that viscosity varies with temperature. In view of their utility in practical situations, in this paper we consider Rayleigh-Bénard convection in three viscoelastic liquids with temperature-dependent viscosity.

2. Mathematical formulation

We consider an infinite horizontal layer of a Jeffreys liquid of thickness d (see Fig. 1). The upper plane at $z=d$ and the lower plane at $z=0$ are maintained at constant temperatures T_0 and $T_0+\Delta T$ respectively. The viscosity μ of the Jeffreys liquid is assumed to depend on the temperature T. In literature various μ-T models have been reported and some of these are recorded below:

Palm-Jenssen model: (Palm [26] and Jenssen [27])

$$\mu(T) = \mu_0\left[1-\delta_1\cos\pi\left\{\frac{1-(T-T_0)}{2}\right\}\right],$$

Torrance-Turcotte model: (Torrance and Turcotte [28], Stengel *et al.* [29])

Table 2
Critical eigen value, wave number and frequency for the oscillatory Rayleigh–Bénard convection in viscoelastic liquid with variable viscosity for different boundary combinations.

Boundary combinations	Λ_1	Q	Pr	$V=0$			$V=0.3$			$V=0.5$		
				R_{oc}	a_c	ω_c	R_{oc}	a_c	ω_c	R_{oc}	a_c	ω_c
FIFI	0.2	0.5	10	561.97 (558.2)[a]	2.503 (2.500)	4.677 (4.619)	495.23 (491.88)	2.504 (2.501)	4.564 (4.506)	462.49 (459.28)	2.503 (2.500)	4.492 (4.434)
	0.3	0.33		364.19 (361.77)	2.489 (2.487)	7.399 (7.356)	320.05 (317.88)	2.489 (2.486)	7.257 (7.216)	298.42 (296.34)	2.487 (2.484)	7.172 (7.131)
		0.4		412.93 (410.02)	2.460 (2.457)	6.310 (6.260)	363.10 (360.49)	2.461 (2.457)	6.188 (6.149)	338.65 (336.16)	2.458 (2.454)	6.119 (6.080)
	0.3	0.4	15	411.07 (408.17)	2.455 (2.452)	6.575 (6.532)	361.06 (358.47)	2.454 (2.450)	6.4906 (6.448)	336.51 (334.03)	2.451 (2.447)	6.438 (6.395)
RIFI/FIRI	0.2	0.5	10	900.48	2.995	5.988	796.05	2.968	5.846	744.44	2.953	5.764
	0.3	0.33		588.34	2.988	8.605	519.00	2.960	8.438	484.76	2.944	8.342
	0.3	0.4		670.12	2.950	7.338	591.29	2.922	7.203	552.36	2.906	7.125
	0.3	0.4	15	668.86	2.948	7.571	589.86	2.919	7.459	550.82	2.904	7.394
RIRI	0.2	0.5	10	1306.24	3.492	7.226	1147.4	3.491	7.157	1069.0	3.501	7.121
	0.3	0.33		858.17	3.491	9.813	752.53	3.495	9.709	700.50	3.499	9.651
	0.3	0.4		981.95	3.444	8.375	861.60	3.447	8.297	802.33	3.45	8.255
	0.3	0.4	15	981.28	3.443	8.578	860.78	3.447	8.525	801.42	3.45	8.591
RAFI/FIRA	0.25	0.4	10	847.55	2.793	4.274	747.27	2.768	4.136	697.77	2.753	4.057
	0.3	0.33		700.06	2.794	5.500	616.62	2.769	5.372	575.43	3.754	5.298
	0.3	0.4		774.26	2.723	4.335	681.83	2.697	4.222	636.21	2.683	4.157
	0.3	0.4	15	771.95	2.718	4.428	679.40	2.692	4.325	633.71	2.677	4.265
RIRA/RARI	0.2	0.5	10	1433.6	3.247	3.556	1259.7	3.251	3.527	1174.2	3.255	3.516
	0.3	0.33		943.21	3.253	6.918	827.45	3.257	6.873	770.52	3.261	6.851
	0.3	0.4		1053.7	3.170	5.612	924.93	3.174	5.580	861.61	3.177	5.565
	0.3	0.4	15	1052.4	3.167	5.701	923.48	3.173	5.685	860.12	3.174	5.674
RIFA/FARI	0.25	0.4	10	618.36	2.793	4.274	545.206	2.768	4.136	509.09	2.753	4.057
	0.3	0.33		510.77	2.794	5.500	449.882	2.769	5.372	419.83	2.754	5.298
	0.3	0.4		564.90	2.723	4.335	497.462	2.697	4.222	464.18	2.683	4.156
	0.3	0.4	15	563.22	2.718	4.428	495.693	2.692	4.325	462.36	2.677	4.265
FAFA	0.3	0.33	10	278.50	2.281	0.429	245.08	2.284	0.354	228.63	2.284	0.164
	0.35	0.29		236.35	2.284	2.6234	207.74	2.288	2.600	193.66	2.288	2.572
	0.35	0.34		253.08	2.203	1.344	222.55	2.205	1.314	207.50	2.204	1.275
	0.35	0.34	15	251.49	2.190	1.347	220.94	2.190	1.320	205.87	2.188	1.282
RAFA/FARA	0.25	0.4	10	591.13	2.700	1.537	520.92	2.676	1.225	486.28	2.662	1.008
	0.3	0.33		488.79	2.704	3.637	430.34	2.680	3.495	401.51	2.667	3.413
	0.3	0.4		530.82	2.611	2.271	467.07	2.587	2.109	435.62	2.573	2.011
	0.3	0.4	15	528.63	2.603	2.311	464.82	3.578	2.153	433.33	2.564	2.057
RARA	0.25	0.4	10	971.46	3.135	4.108	853.17	3.140	4.091	795.03	3.144	4.088
	0.3	0.33		804.76	3.140	5.410	706.25	3.144	5.389	657.82	3.149	5.381
	0.3	0.4		886.15	3.034	4.073	778.07	3.038	4.058	724.97	3.041	4.054
	0.3	0.4	15	884.48	3.030	4.122	776.37	3.033	4.115	723.25	3.036	4.115
FAFI/FIFA	0.25	0.4	10	421.47	2.346	2.293	371.06	2.347	2.226	346.28	2.346	2.173
	0.3	0.33		346.98	2.345	3.974	305.05	2.346	3.915	284.44	2.344	3.872
	0.3	0.4		379.44	2.285	2.889	33.76	2.286	2.837	311.27	2.283	2.798
	0.3	0.4	15	377.01	2.276	2.988	331.26	2.276	2.949	308.74	2.273	2.918

[a] The values within parenthesis are those obtained using the trial function $w=\sin\pi z$, $T=\sin\pi z$.

$$\left.\begin{aligned}\mu(T) &= \mu_0\exp[-\delta_2(T-T_0)],\\ \mu(T) &= \mu_0\exp\left[\delta_3\left(\tfrac{1}{2}-\left(T-T_0+\tfrac{1}{2}\right)\right)\right],\end{aligned}\right\}$$

Nield model: (Nield [30])

$$\mu(T) = \frac{\mu_0}{1+\delta(T-T_0)},$$

Williams-Landel-Ferry (WLF) model: (Peters and Baaijens [31], Wachs and Clermont [32])

$$\mu(T) = \mu_0 a_T(T),$$

$$\lambda_1(T) = \lambda_{10} a_T(T),$$

where $\mu_0 = \mu(T_0), \lambda_{10} = \lambda_1(T_0), a_T(T) = e^{(-c_1(T-T_0))/(c_2+(T-T_0))}$ and c_1 and c_2 are constants.

Straughan model: (Straughan [33] and Siddheshwar [34])

$$\mu(T) = \mu_0\left[1-\delta_4(T-T_0)^2\right].$$

In the above models, T_0 is the reference temperature and δ_i's are small positive constants. Some of these models have been written down in a modified form suitable for the study of Rayleigh-Bénard convection. In all these models, μ_0 refers to the viscosity of the liquid at $T=T_0$. Wachs and Clermont [32] studied a non-isothermal viscoelastic flow using a thermorheological description of WLF. In their model both viscosity and relaxation time depend on temperature. Further a temperature-dependent shift factor, which is non-dimensional, is used in the thermorheological equations.

In the WLF model one may assign values to c_1 and c_2 in having thermorheological coupling that are weak to that which are strong. In our paper, we would have to have one more equation for the strain retardation in the form:

$$\lambda_2(T) = \lambda_{20} a_T(T),$$

where $\lambda_{20} = \lambda_2(T_0)$.

To keep things simpler in the Jeffreys description, the present paper uses the Nield [30] model in studying Rayleigh-Bénard convection rather than the thermodynamically correct WLF model.

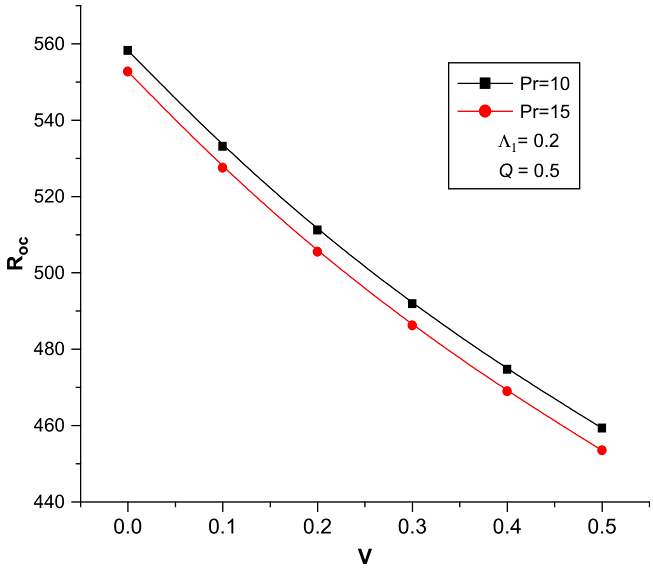

Fig. 2. Plot of R_{oc} vs. V for different values of Pr for FIFI boundaries, for fixed values of Λ_1 and Q.

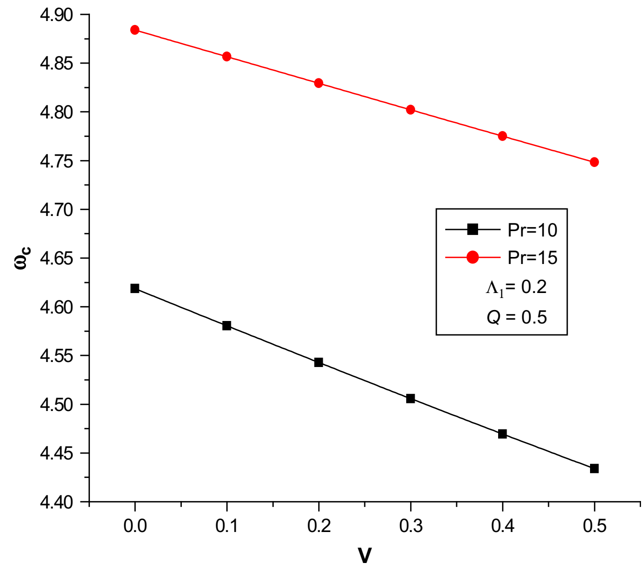

Fig. 4. Plot of ω_c vs. V for different values of Pr for FIFI boundaries, for fixed values of Λ_1 and Q.

The governing equations for the Jeffreys liquid with variable viscosity are

$$q_{i,i} = 0, \tag{1}$$

$$\rho_0 \frac{\partial q_i}{\partial t} = -p_{,i} + \tau'_{ij,j} + \rho g_i, \tag{2}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\tau'_{ij} = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\left[\mu(T)\left(q_{i,j} + q_{j,i}\right)\right], \tag{3}$$

$$\frac{\partial T}{\partial t} + q_i T_{,i} = \kappa T_{,ii}, \tag{4}$$

where $q_i = (u, v, w)$ are the components of the velocity of the liquid, ρ is the density, ρ_0 is the density at the reference temperature T_0, p is the pressure, $\mu(T)$ is the temperature-dependent viscosity of the liquid, $g_i = (0,\ 0,\ -g)$ are the components of the gravitational acceleration, λ_1 is the stress relaxation coefficient, λ_2 is the strain retardation coefficient, T is the temperature and κ is the thermal diffusivity. All the assumptions that lead to the Oberbeck-Boussinesq system are assumed (see Rajagopal *et al.* [35]).

The density equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{5}$$

where $\alpha > 0$ is the constant coefficient of thermal expansion.

The Nield model of shear viscosity as a function of temperature yields the thermorheological equation of state in the form:

$$\mu(T) = \frac{\mu_0}{1 + \delta(T - T_0)}. \tag{6}$$

where $0 < \delta < 1$ and μ_0 is the viscosity at $T = T_0$.

The quiescent basic state has a solution in the form:

$$\left.\begin{aligned}
&q_{ib} = (0,0,0), \quad T = T_b(z) = T_0 + \Delta T\left(1 - \tfrac{z}{d}\right),\\
&\mu = \mu_b(z) = \frac{\mu_0}{[1 + \delta(T_b - T_0)]} = \frac{\mu_0}{\left[1 + \delta\Delta T\left(1 - \frac{z}{d}\right)\right]},\\
&\rho = \rho_b(z) = \rho_0[1 - \alpha(T_b - T_0)] = \rho_0\left[1 - \alpha\Delta T\left(1 - \tfrac{z}{d}\right)\right],\\
&\text{and}\\
&p = p_b(z) = \rho_0 g d\left[\alpha\Delta T\left(\frac{z}{d} - \frac{z^2}{2d^2}\right) - \frac{z}{d}\right] + \text{constant}.
\end{aligned}\right\} \tag{7}$$

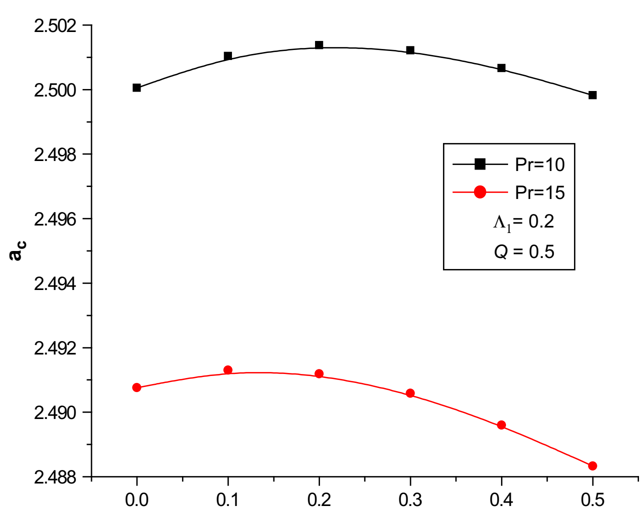

Fig. 3. Plot of a_c vs. V for different values of Pr for FIFI boundaries, for fixed values of Λ_1 and Q.

We now perturb the basic state. Following the classical procedure of linear stability analysis and taking d as the characteristic length, d^2/κ as the characteristic time and ΔT as the characteristic temperature, the linearized dimensionless equations governing small perturbations turn out to be

$$\begin{aligned}
Pr^{-1}\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\nabla^2 w'\right) &= R\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right)\nabla_1^2 T'\\
&+ \left(1 + Q\Lambda_1 \frac{\partial}{\partial t}\right)\left[g(z)\nabla^4 w' + 2Vg^2(z)\nabla^2\left(\frac{\partial w'}{\partial z}\right)\right.\\
&\left. + 2V^2 g^3(z)\left(\frac{\partial^2 w'}{\partial z^2} - \nabla_1^2 w'\right)\right],
\end{aligned} \tag{8}$$

Table 3
Comparison of numerical results for Maxwell fluid, for FIFI boundaries.

Pr	Λ_1	R_{oc}			a_c			ω_c		
		a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
0.1	1.0	416.49	424.90	416.49	3.226	3.228	3.226	6.609	0.964	0.963
1.0	0.1	740.78	740.80	740.79	4.144	4.148	4.144	31.80	8.411	8.390
1.0	1.0	43.386	43.390	43.386	3.290	3.291	3.290	8.004	4.324	4.324
10	1.0	5.9442	4.3480	5.9644	3.788	3.308	3.788	17.37	14.04	15.210

a – Vest and Arpaci [5].
b – Sokolov and Tanner [6].
c – Present work.

$$\frac{\partial T'}{\partial t} - \nabla^2 T' = w', \tag{9}$$

where

$$g(z) = [1 + V(1-z)]^{-1}, \tag{10}$$

$\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2) = \nabla_1^2 + (\partial^2/\partial z^2)$ and primes refer to perturbed quantities.

The dimensionless parameters appearing in equations (8) and (9) are the following:

$$Pr = \frac{\mu_0}{\rho_0 \kappa} \quad \text{(Prandtl number)},$$

$$R = \frac{\alpha \rho_0 g d^3 \Delta T}{\mu_0 \kappa} \quad \text{(Rayleigh number)},$$

$$V = \delta \Delta T \quad \text{(Variable viscosity parameter or thermorheological parameter)},$$

$$\Lambda_1 = \lambda_1 \frac{\kappa}{d^2} \quad \text{(Deborah number or scaled stress-relaxation parameter) and}$$

$$Q = \frac{\lambda_2}{\lambda_1} \quad \text{(Elastic ratio or scaled strain-retardation parameter)}.$$

We assume the perturbations to be periodic waves and hence seek the solution of equations (8) and (9) in the separable form:

$$\left. \begin{aligned} w'(x,y,z,t) &= e^{i\omega t} w(z) e^{i(lx+my)} \\ T'(x,y,z,t) &= e^{i\omega t} T(z) e^{i(lx+my)} \end{aligned} \right\}, \tag{11}$$

where l and m are the horizontal components of wave number and $a^2 = l^2 + m^2$, $w(z)$, $T(z)$ are the amplitudes of the perturbations of velocity and temperature respectively and ω is the frequency.

Substituting equation (11) into equations (8) and (9) and using D to denote the non-dimensional derivative operator (d/dz), we obtain the governing equations for the Jeffreys model in the form:

$$F(\omega)\Big[g(z)\left(D^2 - a^2\right)^2 w + 2Vg^2(z)\left(D^2 - a^2\right) Dw + 2V^2 g^3(z)\left(D^2 + a^2\right) w\Big] - \frac{i\omega}{Pr}\left(D^2 - a^2\right) w - Ra^2 T = 0, \tag{12}$$

$$\left(D^2 - a^2 - i\omega\right) T + w = 0, \tag{13}$$

where $F(\omega) = (1 + iQ\Lambda_1\omega)/(1 + i\Lambda_1\omega)$. We consider ten different boundary combinations in studying Rayleigh-Bénard convection and these are mentioned in Table 1.

We employ the Galerkin technique to compute the critical values of Rayleigh number and wave number. To that end we choose $w(z)$ and $T(z)$ in the form:

$$\left. \begin{aligned} w(z) &= Aw_1(z) \\ T(z) &= BT_1(z) \end{aligned} \right\}, \tag{14}$$

where A and B are constants and w_1 and T_1 are chosen to satisfy the given boundary conditions. The details on these functions are given in Table 1. Multiplying equation (12) by w, (13) by T and integrating with respect to z in [0,1], using equation (14) in the resulting equation we get the following homogeneous equations in A and B:

$$A\left[\frac{x_2}{Pr}\left(i\omega - \Lambda_1\omega^2\right) + (1 + iQ\Lambda_1\omega)x_1\right] - B(1 + i\Lambda_1\omega)a^2 x_3 = 0,$$

$$Ax_3 - B(x_4 + ix_5\omega) = 0,$$

where

Table 4
Comparison of numerical results for Maxwell fluid, for RIRI boundaries.

Pr	Λ_1	R_{oc}			a_c			ω_c		
		a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
0.1	1.0	478.9	480.6	480.61	3.483	3.410	3.415	1.647	1.645	1.64564
1.0	0.1	877.8	894.9	894.87	4.917	4.869	4.873	15.07	14.92	14.9125
1.0	1.0	51.58	51.28	51.279	3.696	3.621	3.61469	6.061	6.061	6.0596
10	0.1	230.0	235.2	235.23	7.309	7.198	7.22005	76.68	75.64	75.8402
10	1.0	7.496	7.521	7.5212	4.724	4.658	4.64617	20.77	20.71	20.6888
100	0.1	130.1	133.9	133.88	11.96	11.75	11.7402	385.8	376.8	376.613
100	1.0	2.203	2.237	2.2373	7.297	7.145	7.15714	83.45	82.24	82.3304
1000	0.1	108.0	112.0	112.05	20.46	19.99	19.9338	2052.0	2006.0	2000.39
1000	1.0	1.289	1.329	1.3287	12.76	11.74	11.7252	418.8	389.0	388.404

a – Vest and Arpaci [5].
b – Sokolov and Tanner [6].
c – Present work.

Table 5
Critical eigen value, wave number and frequency for the oscillatory Rayleigh–Bénard convection in viscoelastic liquid with variable viscosity by considering average temperature T_a as reference temperature for FIFI boundary combination.

Λ_1	Q	Pr	$V=0$			$V=0.1$			$V=0.2$		
			R_{oc}	a_c	ω_c	R_{oc}	a_c	ω_c	R_{oc}	a_c	Ω_c
0.2	0.5	10	558.24	2.500	4.619	558.94	2.500	4.619	561.05	2.500	4.619
0.3	0.33		361.77	2.487	7.356	362.23	2.486	7.357	363.62	2.485	7.359
	0.66		595.56	2.386	2.571	596.30	2.385	2.571	598.54	2.384	2.571
0.2	0.5	15	552.67	2.491	4.884	553.37	2.490	4.884	555.48	2.489	4.883

$$x_1 = \langle g(z)w_1 D^4 w_1\rangle - 2a^2\langle g(z)w_1 D^2 w_1\rangle + a^4\langle g(z)w_1^2\rangle + 2V\langle g^2(z)w_1 D^3 w_1\rangle - 2Va^2\langle g^2(z)w_1 D w_1\rangle + 2V^2\langle g^3(z)w_1 D^2 w_1\rangle + 2V^2a^2\langle g^3(z)w_1^2\rangle,$$

$$x_2 = \langle w_1 D^2 w_1\rangle + a^2\langle w_1^2\rangle, \quad x_3 = \langle w_1 T_1\rangle, \quad x_4 = -\langle T_1 D^2 T_1\rangle + a^2\langle T_1^2\rangle, \quad x_5 = \langle T_1^2\rangle.$$

Now the condition for non-trivial solution of the above homogeneous equations is

$$\begin{vmatrix} \left[\frac{x_2}{Pr}\left(i\omega - \Lambda_1\omega^2\right) + (1 + iQ\Lambda_1\omega)x_1\right] & -(1 + i\Lambda_1\omega)a^2x_3 \\ x_3 & -(x_4 + ix_5\omega) \end{vmatrix} = 0.$$

Solving, we get the following expression for the eigen value:

$$R = \frac{x_1x_5\omega^2N_2 + x_1x_4N_1 - \frac{x_2x_5\omega^2}{Pr}}{a^2x_3^2} + i\omega\frac{N_3}{a^2x_3^2}, \tag{15}$$

where

$$N_1 = \frac{1 + \Lambda_1^2Q\omega^2}{1 + \Lambda_1^2\omega^2}, \quad N_2 = \frac{\Lambda_1(1 - Q)}{1 + \Lambda_1^2\omega^2},$$
$$N_3 = x_1x_5N_1 - x_1x_4N_2 + \frac{x_2x_4}{Pr}.$$

The eigen value R, given by equation (15), is real and therefore the imaginary part of the equation (15) must be zero. We have here two possibilities: either $\omega = 0$ ($N_3 \neq 0$) or $N_3 = 0$ ($\omega \neq 0$).

Stationary Instability (when $\omega = 0$ and $N_3 \neq 0$):
Equation (15) in this case is given by

$$R_s = \frac{x_1x_4N_1}{a^2x_3^2}. \tag{16}$$

Oscillatory Instability (when $\omega \neq 0$ and $N_3 = 0$):
Equation (15) in this case is given by

$$R_o = \frac{x_1x_5\omega^2N_2 + x_1x_4N_1 - \frac{x_2x_5\omega^2}{Pr}}{a^2x_3^2}, \tag{17}$$

where ω^2 can be obtained by taking $N_3 = 0$ and this gives us

$$\omega^2 = -\frac{Prx_1[\Lambda_1(Q - 1)x_4 - x_5] + x_2x_4}{\Lambda_1^2[x_2x_4 + PrQx_1x_5]}. \tag{18}$$

The following table documents the validity or otherwise of the principle of exchange of stabilities in the three different liquids studied in the paper.

Q	Type of liquid	Principle of exchange of stabilities
∞	Rivlin–Ericksen	Valid
0	Maxwell	Not valid
$0 < Q < 1$	Jeffreys	Not valid

The validity of the principle of exchange of stabilities in the case of Rivlin-Ericksen liquid is proved in the section on "Results and discussion". We now move over to the discussion of the computed results obtained by the Galerkin technique for all ten boundary combinations.

3. Results and discussion

The paper deals with a linear stability analysis of Rayleigh-Bénard convection in Maxwell, Rivlin-Ericksen and Jeffreys liquids. In our calculations we have assumed value of Prandtl number larger than that assumed in Newtonian liquids due to the fact that the magnitude of the dynamic viscosity in viscoelastic liquids is much greater than that of the Newtonian liquids. We now discuss the results on Rayleigh-Bénard convection that manifests via the oscillatory mode.

From equation (18) it is clear that the stationary convection is possible when $Q > 1$ or when $Q < 1$ for some range of parameters. Oscillatory convection may be possible only when $Q < 1$. This is true for all ten boundary combinations considered, and is also true in the case of constant viscosity (see Siddheshwar, [18]).

The critical value of the wave number a_c and the frequency ω_c, which in turn give the critical Rayleigh number R_c for the onset of convection, were computed for various parameter ranges and the results are presented in Table 2. From the table it is clear that the difference between the results of single term and higher order is less than 1% for all types of viscoelastic liquids considered. From the table it is also clear that in the case of oscillatory

Table 6
Comparison of critical eigen value, wave number and frequency for the oscillatory Rayleigh–Bénard convection in a viscoelastic liquid with variable viscosity for two different viscosity models for fixed values of Λ_1, Q and Pr.

	a			b		
V	R_{oc}	a_c	ω_c	R_{oc}	a_c	ω_c
$\Lambda_1 = 0.2$, $Q = 0.5$, $Pr = 10$						
0	558.24	2.500	4.619	558.24	2.500	4.619
0.1	532.20	2.501	4.579	533.09	2.501	4.580
0.2	508.04	2.502	4.538	511.18	2.501	4.543
0.3	485.63	2.503	4.496	491.88	2.501	4.506
0.4	464.82	2.503	4.453	474.71	2.501	4.469
0.5	445.48	2.503	4.408	459.28	2.500	4.434
$\Lambda_1 = 0.3$, $Q = 0.33$, $Pr = 10$						
0	361.77	2.487	7.356	361.77	2.487	7.356
0.1	344.54	2.487	7.306	345.13	2.487	7.308
0.2	328.56	2.488	7.255	330.64	2.487	7.261
0.3	313.75	2.487	7.203	317.88	2.486	7.216
0.4	300.00	2.487	7.150	306.53	2.485	7.172
0.5	287.23	2.486	7.097	296.34	2.484	7.131

a – Exponential model ([28,29]).
b – Nield model ([30]).

convection the effect of increasing Λ_1 and Pr is to decrease R_{oc} for Maxwell and Jeffreys liquids. Effect of increasing Q is to increase R_{oc}. We also find that R_{oc} decreases with increasing V. Figs. 2-4 have been plotted to explicitly demonstrate the fact that the effect of increasing Pr on a_c and ω_c is more pronounced than that on R_{oc}.

In the case of Rivlin-Ericksen liquid that subscribes to the limitations discussed by Dunn and Rajagopal [36], oscillatory convection is not possible. This can be shown as follows:

For Rivlin-Ericksen liquid, $\Lambda_1 = 0$ and hence R given by equation (14) takes the form:

$$R = \frac{x_1x_4 - \frac{x_2x_5\omega^2}{Pr}}{a^2x_3^2} + i\omega\frac{\left(x_1x_5 + \frac{x_2x_4}{Pr}\right)}{a^2x_3^2}.$$

Clearly for R to be real the only possibility is $\omega = 0$, as can be seen quite easily from the imaginary part of R. Hence, we can conclude that the principle of exchange of stabilities is valid in the case of Rivlin-Ericksen liquid. Thus the Rivlin-Ericksen liquid behaves like a Newtonian liquid while in a Rayleigh-Bénard situation.

The results of the paper are validated by comparing them with the related results of previous works and this is given in Tables 3 and 4. From Table 3 we observe that our results agree quite closely with the works of Sokolov and Tanner [6] for Maxwell liquids, for FIFI boundaries. There is, however, a mismatch between our results and those of Vest and Arpaci [5] due to the fact that there is a mistake in the expression of ω^2 as reported by them. From Table 4 we observe that our results also agree quite closely with those of Vest and Arpaci [5] and Sokolov and Tanner [6] for Maxwell liquids, for RIRI boundaries.

For most liquids the curve representing the temperature-dependence of viscosity is known to be a decreasing function and is concave upwards. In the case of quadratic dependency of viscosity on temperature, as used by Straughan [33] and Siddheshwar [34], the curve is concave downwards. The exponential ([28,29]) and Nield [30] models aptly describe the thermorheological description. A comparison is made for the critical values obtained by these two models and the results are presented in Table 5. Clearly the exponential ([28,29]) and Nield [30] models yield results that match quite well. Therefore, in this paper Nield model [30] is considered to obtain the critical values for different boundary combinations.

We now move over to the discussion on some general results.

General results

(a) Comparing results of Rivlin-Ericksen (and Newtonian), Maxwell and Jeffreys liquids the following observation is true:

$$R_{sc}^{\text{Newtonian}} = R_{sc}^{\text{Rivlin-Ericksen}} > R_{oc}^{\text{Jeffreys}} > R_{oc}^{\text{Maxwell}},$$
$$a_c^{\text{Maxwell}} > a_c^{\text{Jeffreys}} > a_c^{\text{Rivlin-Ericksen}} = a_c^{\text{Newtonian}},$$
$$\omega_c^{\text{Maxwell}} > \omega_c^{\text{Jeffreys}}.$$

(b) Comparing results of all the considered boundary combinations, the following observation is true in the case of Maxwell and Jeffreys liquids:

$$R_{oc}^{RIRA} > R_{oc}^{RIRI} > R_{oc}^{RARA} > R_{oc}^{RAFI} > R_{oc}^{RIFI} > R_{oc}^{RIFA} > R_{oc}^{RAFA} > R_{oc}^{FIFI} > R_{oc}^{FAFI} > R_{oc}^{FAFA},$$
$$a_c^{RIRI} > a_c^{RIRA} > a_c^{RARA} > a_c^{RIFI} > a_c^{RIFA} = a_c^{RAFI} > a_c^{RAFA} > a_c^{FIFI} > a_c^{FAFI} > a_c^{FAFA},$$
$$\omega_c^{RIRI} > \omega_c^{RIFI} > \omega_c^{FIFI} > \omega_c^{RIRA} > \omega_c^{RIFA} = \omega_c^{RAFI} > \omega_c^{RARA} > \omega_c^{FAFI} > \omega_c^{RAFA} > \omega_c^{FAFA}.$$

(c) Comparing results of Maxwell and Jeffreys liquids, we find that the following observations are true:
- a_c increases with increase in V for both rigid boundaries, decreases for rigid-free boundaries and increases and then decreases for both free boundary combinations.
- ω_c decreases with increase in V.

4. Conclusions

The results indicate that the effect of variable viscosity parameter is to destabilize the system. The results pertaining to Jeffreys liquid lead to those of Maxwell, Rivlin-Ericksen (and Newtonian liquids) by suitable limiting processes. It is observed that Maxwell liquids are more unstable than Jeffreys liquids whereas Rivlin-Ericksen (and Newtonian liquids) are comparatively more stable. It is clear from the obtained results that the Rivlin-Ericksen liquid behaves like a Newtonian liquid when in a Rayleigh-Bénard situation. This is true both in the absence and presence of temperature-dependence of viscosity.

At this point we note that our conclusions are based on the choice of T_0 (temperature of the upper boundary) as the reference temperature. As pointed out by Nield [30] the study can be carried out with $T_a = ((T_0 + \Delta T) + T_0)/2$ (average temperature of the two boundaries) as the reference temperature. With T_0 replaced by T_a in the above analysis we conclude the following in respect of FIFI boundary combination:

(1) R_c increases as V increases but the variation is very weak as pointed out by Nield [30].
(2) a_c decreases as V increases but only in the third decimal digit.
(3) ω_c remains invariant when V varies.
(4) The effect of Λ_1, Q and Pr on R_c, a_c and ω_c is same as that is seen when T_0 is used.
(5) Exponential ([28,29]) and Nield [30] models yield results that match quite well.

The conclusions for FIFI boundary combination are drawn from Table 6 and on computation we find that for other boundary combinations also these conclusions are true as seen by Vanishree and Siddheshwar [37].

Acknowledgement

The work reported in the paper is carried out at the VTU research center in the Department of Mathematics, BMS College of Engineering, Bangalore. The authors express their gratitude to the managements of the respective institutions for the encouragement received for their work. The authors wish to thank Dr. Pradeep G. Siddheshwar for the suggestions and the anonymous reviewers for many helpful comments.

References

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, Oxford, 1961.

[2] J.K. Platten, J.C. Legros, Convection in Liquids, Springer, Berlin, 1984.
[3] D.M. Herbert, On the stability of viscoelastic liquids in heated plane Couette flow, J. Fluid Mech. 17 (1963) 353–359.
[4] T. Green, Oscillating convection in an elasticoviscous liquid, Phys. Fluid. 11 (1968) 1410–1413.
[5] C.M. Vest, V.S. Arpaci, Overstability of a viscoelastic fluid layer heated from below, J. Fluid Mech. 36 (1969) 613–623.
[6] M. Sokolov, R.I. Tanner, Convective stability of a general viscoelastic fluid heated from below, Phys. Fluid. 15 (1972) 534–539.
[7] N. Riahi, Nonlinear convection in a viscoelastic fluid, J. Math. Phys. Sci. 26 (1976) 211–220.
[8] I.A. Eltayeb, Nonlinear thermal convection in an elasticoviscous layer heated from below, Proc. Roy. Soc. Lond. A 356 (1977) 161–176.
[9] A.V. Shenoy, R.A. Mashelkar, Thermal convection in non-Newtonian fluids, Adv. Heat Transf. 15 (1982) 143–225.
[10] S. Rosenblat, Thermal convection in a viscoelastic liquid, J. Non-Newtonian Fluid Mech. 21 (1986) 201–223.
[11] R.W. Kolkka, G.R. Ierley, On the convected linear stability of a viscoelastic Oldroyd-B fluid heated from below, J. Non-Newtonian Fluid Mech. 25 (1987) 209–237.
[12] R.E. Khayat, Chaos and overstability in the thermal convection of viscoelastic fluids, J. Non-Newtonian Fluid Mech. 53 (1994) 227–255.
[13] R.E. Khayat, Nonlinear overstability in the thermal convection of viscoelastic fluids, J. Non-Newtonian Fluid Mech. 58 (1995) 331–356.
[14] H.M. Park, H.S. Lee, Hopf bifurcation of viscoelastic fluid heated from below, J. Non-Newtonian Fluid Mech. 66 (1996) 1–34.
[15] J.M. Mardones, R. Tiemann, D. Walgraef, Thermal convection thresholds in viscoelastic solutions, J. Non-Newtonian Fluid Mech. 93 (2000) 1–15.
[16] P.G. Siddheshwar, C.V. Srikrishna, Rayleigh-Bénard convection in a viscoelastic fluid-filled high-porosity medium with nonuniform basic temperature gradient, Int. J. Math. Math. Sci. 25 (2001) 609–619.
[17] H.M. Park, D.H. Ryu, Rayleigh-Bénard convection of viscoelastic fluids in finite domains, J. Non-Newtonian Fluid Mech. 98 (2001) 169–184.
[18] P.G. Siddheshwar, Oscillatory convection in ferromagnetic, dielectric and viscoelastic liquids, Int. J. Mod. Phys. B. 16 (2002) 2629–2635.
[19] P.G. Siddheshwar, C.V. Srikrishna, Unsteady nonlinear convection in second order fluid, Int. J. Non Lin. Mech. 37 (2002) 321–330.
[20] E. Abu-Ramadan, J.M. Hay, R.E. Khayat, Characterization of chaotic thermal convection of viscoelastic fluids, J. Non-Newtonian Fluid Mech. 115 (2003) 79–113.
[21] M.C. Kim, S.B. Lee, S. Kim, B.J. Chung, Thermal instability of viscoelastic fluids in porous media, Int. J. Heat Mass Transf. 46 (2003) 5065–5072.
[22] H.M. Park, K.S. Park, Rayleigh-Bénard convection of viscoelastic fluids in arbitrary finite domains, Int. J. Heat Mass Transf. 47 (2004) 2251–2259.
[23] D.Y. Yoon, M.C. Kim, C.K. Choi, The onset of oscillatory convection in a horizontal porous layer saturated with viscoelastic liquid, Transport Porous Media 55 (2004) 275–284.
[24] V. Bertola, E. Cafaro, Thermal instability of viscoelastic fluids in horizontal porous layers as initial value problem, Int. J. Heat Mass Transf. 49 (2006) 4003–4012.
[25] P.G. Siddheshwar, C.V. Srikrishna, Effect of time-periodic modulation on convection in solutions of dilute polymeric liquids in a porous medium, Int. J. Appl. Mech. Eng. 13 (2008) 753–780.
[26] E. Palm, On the tendency towards hexagonal cells in steady convection, J. Fluid Mech. 8 (1960) 183–192.
[27] O. Jenssen, Note on the influence of variable viscosity on the critical Rayleigh number, Acta Polytech. Scand. 24 (1963) 1–7.
[28] K.E. Torrance, D.L. Turcotte, Thermal convection with large viscosity variations, J. Fluid Mech. 47 (1971) 113–125.
[29] K.C. Stengel, D.S. Oliver, J.R. Booker, Onset of convection in a variable-viscosity fluid, J. Fluid Mech. 120 (1982) 411–431.
[30] D.A. Nield, The effect of temperature-dependent viscosity on the onset of convection in a saturated porous medium, J. Heat Transf. 118 (1996) 803–805.
[31] G.W.M. Peters, F.P.T. Baaijens, Modelling of non-isothermal viscoelastic flows, J. Non-Newtonian Fluid Mech. 68 (1997) 205–224.
[32] A. Wachs, J.-R. Clermont, Non-isothermal viscoelastic flow computations in an axisymmetric contraction at high Weissenberg numbers by a finite volume method, J. Non-Newtonian Fluid Mech. 95 (2000) 147–184.
[33] B. Straughan, The Energy Method, Stability and Nonlinear Convection, Springer, New York, 2004.
[34] P.G. Siddheshwar, Thermorheological effect on magnetoconvection in weak electrically conducting fluids under 1g or μg, Pramana J. Phys. (India) 62 (2004) 61–68.
[35] K.R. Rajagopal, M. Ruzicka, A.R. Srinivas, On Oberbeck-Boussinesq approximation, Math. Model. Meth. Appl. Sci. 6 (1996) 1157–1167.
[36] J.E. Dunn, K.R. Rajagopal, Fluids of differential type, critical review and thermodynamic analysis, Int. J. Eng. Sci. 33 (1995) 689–729.
[37] R.K. Vanishree, P.G. Siddheshwar, Effect of rotation on thermal convection in an anisotropic porous medium with temperature-dependent viscosity. Transport Porous Media, in press.